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Manifolds and Group Actions

Homework 2

Mandatory Exercise 1. (5 points)
Show that the definition of a differentiable map does not depend on the choice of the parametrizations.

Mandatory Exercise 2. (5 points)
Show that the antipodal map $f : S^n \to S^n$ defined by $f(x) = -x$ is a differentiable map.

Mandatory Exercise 3. (10 points)
Let M, N and P be smooth manifolds.

(a) Consider the identity map $f\colon M \to M$. Show that $df\colon TM \to TM$ is also the identity map.

(b) Let $f\colon M \to N$ and $g\colon N \to P$ be two differentiable maps. Show that $g \circ f\colon M \to P$ is also differentiable and that

$$\big(d(g \circ f)\big)_p = (dg)_{f(p)} \circ (df)_p,$$

holds for all $p \in M$.

(c) If $f\colon M \to N$ is a diffeomorphism, then $df\colon TM \to TN$ is also bijective with inverse map given by $d(f^{-1})$.

Suggested Exercise 1. (0 points)
Consider the two atlases $\mathcal{A}_1 = \{(\mathbb{R}, \varphi_1)\}$ and $\mathcal{A}_2 = \{(\mathbb{R}, \varphi_2)\}$ on $\mathbb{R}$ given by $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$.

(a) Show that $\varphi_2^{-1} \circ \varphi_1$ is not differentiable and conclude that the two atlases are not equivalent.

(b) The identity map $\mathrm{id}\colon \{(\mathbb{R}, \varphi_1)\} \to \{(\mathbb{R}, \varphi_2)\}$ is not a diffeomorphism.

(c) The map $f\colon \{(\mathbb{R}, \varphi_1)\} \to \{(\mathbb{R}, \varphi_2)\}$ given by $f(x) = x^3$ is a diffeomorphism. Conclude that the two differentiable structures are diffeomorphic.

Suggested Exercise 2. (0 points)
Let $\{(U_\alpha, \varphi_\alpha)\}$ be a differentiable structure on M and consider the maps

$$\begin{aligned}\Phi_\alpha \colon U_\alpha \times \mathbb{R}^n &\longrightarrow TM \\ (x, v) &\longmapsto (d\varphi_\alpha)_x(v) \in T_{\varphi_\alpha(x)}M.\end{aligned}$$

(a) Show that the family $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$ is a smooth atlas.

(b) Conclude that TM carries the structure of a differentiable manifold. What dimension has TM?

(c) If $f \colon M \to N$ is differentiable, then $df \colon TM \to TN$ is also differentiable.

Suggested Exercise 3. (0 points)
Let M be an n-dimensional differentiable manifold and $p \in M$. Show that the following set can be canonically identified with T_pM (and therefore constitute an alternative geometric definition of the tangent space):
$C_p/_\sim$, where C_p is the set of differentiable curves $c \colon I \to M$ such that $c(0) = p$ and $\sim$ is the equivalence relation defined by

$$c_1 \sim c_2 :\Leftrightarrow \frac{d}{dt}\big(\varphi^{-1} \circ c_1\big)(0) = \frac{d}{dt}\big(\varphi^{-1} \circ c_2\big)(0)$$

for some parametrization $\varphi \colon U \to M$ of a neighborhood of p.

Suggested Exercise 4. (0 points)
The **connected sum** of two topological n-manifolds M and N is the topological manifold $M\#N$ obtained as follows. One chooses subsets $U \subset M$ and $V \subset N$ homeomorphic to closed balls and a homeomorphism $h : \partial U \to \partial V$. One then defines the smallest equivalence relation $\sim$ on

$$M \setminus \operatorname{int}(U) \cup N \setminus \operatorname{int}(V),$$

which satisfies $x \sim h(x)$ for all $x \in \partial U$. We are basically removing open balls from the manifolds and gluing the resulting manifolds along their boundaries. (In principle this manifold might depend on the choice of homeomorphism and balls, which we do not worry about here.)

(a) Give examples of this construction.

(b) Show that $M\#S^n$ is homeomorphic to M.

(c) Show that $T^2\#\mathbb{R}P^2$ is homeomorphic to $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$.

Hand in: Tuesday 2nd of May in the exercise session